L08 Hypothesis testing

1. An example of hypothesis testing
   A machine is set up to produce a product with the mean width 3 mm. To test the setting of the machine a sample of 50 products is collected, and a test is conducted.
   For the width $X \sim N(\mu, \sigma^2)$ with known σ,

$H_0: \mu = 3$ versus $H_a: \mu \neq 3$
Test statistic: $T = \frac{\overline{X}-3}{\sigma/\sqrt{n}}$
Reject H_0 if $T < -1.96$ or $T > 1.96$ for $\alpha = 0.05$
$Z_{ob} = 0.0032$.
Fail to reject H_0. The incorrect setting was not detected.

   (1) Three-step test scheme
   To specify a test scheme one must specify (i) the null and the alternative hypotheses (ii) the test statistic (iii) the decision rule

   (2) Two-step implementation
   To implement a test after specifying the test scheme there are (iv) computations and (v) conclusion.

2. Analysing a test

   (1) Type I error and Type II error

	$\theta \in H_0$	$\theta \in H_a$
Rejecting H_0	Type I error	
Accepting H_0		Type II error

   (2) Probability of rejecting H_0
   For a test scheme, $\beta(\theta) = P_\theta(\text{Rejecting } H_0)$ is a function of θ

   (3) Probability of Type I error, Power and Probability of Type II error

Probability	$\theta \in H_0$	$\theta \in H_a$
Rejecting H_0	$\beta(\theta) = P(\text{Type I error})$	$\beta(\theta)$: Power of the test at θ
Accepting H_0	$1 - \beta(\theta)$	$1 - \beta(\theta) = P(\text{Type II error})$

   (4) α-level test

   $$\begin{aligned}\text{A test is an } \alpha\text{-level test} &\overset{def}{\iff} \beta(\theta) \leq \alpha \text{ for all } \theta \in H_0 \\ &\iff P(\text{Type I error}) \leq \alpha \text{ at all } \theta \in H_0 \\ &\iff \sup\,[\beta(\theta) : \theta \in H_0] \leq \alpha\end{aligned}$$

   Here the pre-selected α is called the significance of the test.

   (5) Unbiased test

   $$\begin{aligned}\text{A test is unbiased} &\overset{def}{\iff} \beta(\theta_1) \leq \beta(\theta_2) \text{ for all } \theta_1 \in H_0 \text{ and all } \theta_2 \in H_a \\ &\iff \sup\,[\beta(\theta) : \theta \in H_0] \leq \beta(\theta) \text{ for all } \theta \in H_a \\ &\iff \text{Significance level } \alpha \leq \beta(\theta) \text{ for all } \theta \in H_a.\end{aligned}$$

   For an unbiased test the probability of rejecting H_0 when H_0 is true is always less than or equal to the probability of rejecting H_0 when H_0 is false, i.e., $P(\text{Type I error}) \leq$Power.

   (6) Comparison of powers
   Suppose T and T_* are two tests on the same hypotheses.

T is more powerful than T_* at $\theta_1 \overset{def}{\Longleftrightarrow} \theta_1 \in H_a$ and $\beta_T(\theta_1) \geq \beta_{T_*}(\theta_1)$.

T is uniformly more powerful than $T_* \overset{def}{\Longleftrightarrow} \beta_T(\theta) \geq \beta_{T_*}(\theta)$ for all $\theta \in H_a$

(7) Uniformly most powerful (UMP) test in a class

T is UMP test in a class $\overset{def}{\Longleftrightarrow}$ T is uniformly more powerful than all tests in the class.

Comment: For given hypothesis we always try to construct an α-level test and check if the test is unbiased or even if UMP test in the class.

3. Example

Consider the test in 1.

(1) Find the expression for $\beta(\mu)$

$X \sim N(\mu,\, \sigma^2) \Longrightarrow \overline{X} \sim N\left(\mu,\, \frac{\sigma^2}{n}\right) \Longrightarrow T = \frac{\overline{X}-3}{\sigma/\sqrt{n}} \sim N\left(\frac{\mu-3}{\sigma/\sqrt{n}},\, 1^2\right) \sim \frac{\mu-3}{\sigma/\sqrt{n}} + N(0,\, 1^2)$.

$$\begin{aligned}
\beta(\mu) &= P(T < -1.96) + P(T > 1.96) \\
&= P\left(N(0,\, 1^2) < -1.96 - \tfrac{\mu-3}{\sigma/\sqrt{n}}\right) + P\left(N(0,\, 1^2) > 1.96 - \tfrac{\mu-3}{\sigma/\sqrt{n}}\right)
\end{aligned}$$

(2) Show that the test has level $\alpha = 0.05$

$$\begin{aligned}
\alpha &= \sup\left[\beta(\mu) : \, \mu \in H_0\right] = \beta(3) \\
&= P(N(0,\, 1^2) < -1.96 - 0) + P(N(0,\, 1^2) > 1.96 - 0) \\
&= 0.025 + 0.025 = 0.05.
\end{aligned}$$

(3) Show that the test is unbiased.

Need to show $\alpha = 0.05 \leq \beta(\mu)$ for all $\mu \neq 3$, i.e.,

(i) $\mu < 3 \Longrightarrow \beta(\mu) \geq 0.05$ (ii) $\mu > 3 \Longrightarrow \beta(\mu) \geq 0.05$.

We show (i) only.

$\mu < 3 \Longrightarrow \frac{\mu-3}{\sigma/\sqrt{n}} < 0 \Longrightarrow h = -\frac{\mu-3}{\sigma/\sqrt{n}} > 0$.

$$\begin{aligned}
\beta(\mu) - 0.05 &= [P(N(0,\, 1^2) < -1.96 + h) + P(N(0,\, 1^2) > 1.96 + h)] \\
&\quad -[P(N(0,\, 1^2) < -1.96) + P(N(0,\, 1^2) > 1.96)] \\
&= P(-1.96 < Z < -1.96 + h) - P(1.96 < Z < 1.96 + h) \\
&= P(1.96 - h < Z < 1.96) - P(1.96 < Z < 1.96 + h) \geq 0
\end{aligned}$$